THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Feb 7

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

Part I: Solutions to additional exercises

1. Think about following statements and determine whether they are true or false:

   (a) **(Theorem 3.2.10)** Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim_{n\to\infty}(\sqrt{x_n}) = \sqrt{x}$.

   (b) Let $X = (x_n^2)$ be a sequence of real numbers that converges to x and suppose that $x \geq 0$. Then the sequence x_n converges and $\lim_{n\to\infty} x_n = \sqrt{x}$.

   (c) Let $X = (x_n^2)$ be a sequence of real numbers that converges to $x = 0$. Then the sequence x_n converges and $\lim_{n\to\infty} x_n = 0$.

   (**Notice that in (b), (c) we are not assuming** $x_n \geq 0$)

   Solutions:

   (a) True. Refer to the proof on page 68 of the textbook.

   (b) False. Consider $(x_n) = (1, -1, 1, -1, 1, -1, \cdots)$. Then $X = (x_n^2) = (1, 1, 1, \cdots)$ converges to $x = 1 \geq 0$ while (x_n) is not convergent.

   (c) True. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n^2 - 0| = |x_n|^2 < \varepsilon^2$.
   Therefore, $\forall n \geq N$, $|x_n| < \varepsilon$ and thus $\lim_{n\to\infty} x_n = 0$.

   Remark: The difference between (a) and (b) is that in (b) we do not assume (x_n) is a sequence of non-negative real numbers. So that (x_n) can be an alternative sequence, i.e., the terms have alternative signs while their absolute values approach x meanwhile.

   But things are different if the limit $x = 0$. In this case (x_n) can still be oscillating but the terms are within ε of 0.

   As a summary, we have

$$\lim_{n\to\infty} x_n^2 = x \iff \lim_{n\to\infty} |x_n| = \sqrt{x} \not\Rightarrow \lim x_n = \sqrt{x},$$
$$\lim x_n = \sqrt{x} \Rightarrow \lim_{n\to\infty} x_n^2 = x \iff \lim_{n\to\infty} |x_n| = \sqrt{x}.$$

2. **(Average of a sequence).** Let (x_n) be any sequence of real numbers. We define its partial sum by

$$S_n = \sum_{k=1}^{n} x_k,$$

   and the average of it by

$$A_n = \frac{S_n}{n}.$$

(a) Show that if $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then

$$\lim_{n\to\infty} A_n = x.$$

(b) Show that the converse is not true by giving a counterexample, i.e., a real sequence (x_n) whose average converges to a finite limit $L \in \mathbb{R}$ but x_n itself does not.

Solutions:

(a) It suffices to prove the conclusion for the case that $x = 0$. We desire to show that $\dfrac{|x_1 + x_2 + \cdots + x_n|}{n}$ can be arbitrarily close to 0 on condition that $\lim_{n\to\infty} x_n = 0$. The idea is to split the sum $x_1 + x_2 + \cdots + x_n$ into two parts. One part consists of finite terms so their sum is a fixed constant and the quotient can be as small as we want when divided by a natural number n that is large enough, while in the other part every term is close enough to 0.

Write above arguments in explicit mathematical language: from $\lim_{n\to\infty} x_n = 0$ we have

$$\forall \varepsilon > 0, \quad \exists N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1, |x_n| < \frac{\varepsilon}{2}.$$

Moreover, by Archimedean Property, there exists $N_2 \in \mathbb{N}$ such that

$$N_2 > \frac{2|x_1 + x_2 + \cdots + x_{N_1}|}{\varepsilon}.$$

Then for any $n \geq N := \max(N_1, N_2)$, we have

$$\begin{aligned}
|A_n| = \frac{|x_1 + x_2 + \cdots + x_n|}{n} &= \frac{|x_1 + x_2 + \cdots + x_{N_1} + x_{N_1+1} + \cdots + x_n|}{n} \\
&\leq \frac{|x_1 + x_2 + \cdots + x_{N_1}| + |x_{N_1+1}| + \cdots + |x_n|}{n} \\
&\leq \frac{|x_1 + x_2 + \cdots + x_{N_1}|}{N_2} + \frac{n - N_1}{n} \cdot \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} A_n = 0$.

For general case, we define another sequence (y_n) by $y_n = x_n - x$. Then $\lim_{n\to\infty} y_n = 0$ and from previous argument we have

$$\lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = 0$$

which implies

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x + \lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = x.$$

(b) Consider $x_n = (-1)^n$. Then

$$S_n = \frac{(-1)^n - 1}{2} \Longrightarrow A_n = \frac{(-1)^n - 1}{2n}.$$

By Squeeze Theorem we know $\lim_{n\to\infty} A_n = 0$. However, it's obvious that (x_n) is divergent.

Part II: Other problems

1. **(Limit theorems)**. Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively, and let $c \in \mathbb{R}$. Then the sequences $X + Y, X - Y, X \cdot Y, cX$ converge to $x + y, x - y, xy, cx$ respectively.

   Think about the following statements and determine whether they are true or false.

   (a) If X converges to x and Y is divergent, then $X + Y$ is divergent.
   (b) If X converges to x and Y is divergent, then $X \cdot Y$ is divergent.
   (c) If X is divergent, then cX is divergent.
   (d) If both X and Y are divergent, then $X + Y$ is divergent.
   (e) If both X and Y are divergent, then $X \cdot Y$ is divergent.

   Answers:

   (a) True. Otherwise $Y = (X + Y) + (-X)$ would be convergent.
   (b) False. Consider $X = (0, 0, 0, 0, \cdots)$.
   (c) False. Consider $c = 0$.
   (d) False. Consider $Y = -X$.
   (e) False. Consider $X = (0, 1, 0, 1, \cdots), Y = (1, 0, 1, 0, \cdots)$.

2. **(Comparison of order of growth).** We have learned a lot about the growth rate of different kinds of sequences. For example, $\lim_{n\to\infty} \frac{n}{2^n} = 0$, which can be understood as 2^n grows faster than n, as they both tends to infinity. Let's look at more results:

$$1 \ll n \ll n^2 \ll n^{100} \ll 2^n \ll 100^n \ll n! \ll n^n.$$

   Here, $a_n \ll b_n$ means **(we only use this notation in tutorial classes)**

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

   I will show $n! \ll n^n$ without using ratio test:

   If $n > 2K$ then (we can take $K = \left[\frac{n-1}{2}\right]$)

$$\begin{aligned}\frac{n!}{n^n} &= \frac{1 \cdot 2 \cdots (K-1) \cdot K}{n^K} \cdot \frac{(K+1) \cdots (n-1)n}{n^{n-K}} \\ &\leq \frac{1 \cdot 2 \cdots (K-1)K}{n^K}\end{aligned}$$

$$< \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \left(\frac{1}{2}\right)^K.$$

So we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (by Squeeze theorem or Theorem 3.1.10).

Also, for $b^n \ll n!, \forall b > 1$: when $n > [b] + 1 := B$, we have

$$\begin{aligned}\frac{b^n}{n!} &= \frac{b}{1} \cdot \frac{b}{2} \cdots \frac{b}{B} \cdot \frac{b}{B+1} \cdots \frac{b}{n} \leq \frac{b}{1} \cdot \frac{b}{2} \cdots \frac{b}{B} \cdot \frac{b}{B} \cdots \frac{b}{B} \\ &= \frac{b}{1} \cdot \frac{b}{2} \cdots \frac{b}{B-1} \cdot \left(\frac{b}{B}\right)^{n-B+1}.\end{aligned}$$

Since $0 < b/B < 1$ from our definition of B, we conclude that $\lim_{n\to\infty} \frac{b^n}{n!} = 0$.

As an exercise, you may try to prove the remaining unsolved case:

$$n^a \ll b^n, \forall a > 0, b > 1.$$

3. **(Ratio test).** Let (x_n) be a sequence of positive real numbers and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$$

where L is a non-negative real number.

(a) If $0 \leq L < 1$, then $\lim_{n\to\infty} x_n = 0$.

(b) If $L > 1$, then (x_n) is divergent.

(c) (**Ex 3.2.17**) If $L = 1$, then (x_n) can be either divergent or convergent, i.e., this method fails.

Examples.

(a) Consider the sequence in Problem 1 again where $x_n = \frac{n!}{n^n}$ and we have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1} \in (0,1)$$

and thus we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

As an exercise to Section 3.3, you can try to prove the above limit in red.

(c) Consider the following two sequences respectively:

i. $x_n = n$.

ii. $x_n = \frac{1}{n}$.

Remark: Similarly we have the **root test** if we define

$$\lim_{n\to\infty} \sqrt[n]{x_n} = L.$$

You can refer to **Exercises 3.2.20-21** in the textbook.

Part III: Some comments

1. We have learned the limit of a sequence. As I said in the first tutorial, the definitions are very important and every word should be accurate and precise. Let's look at the statement: A sequence $X = (x_n)$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > K(\varepsilon), |x_n - x| < \varepsilon.$$

Keywords: $\forall \varepsilon > 0, \forall n > K(\varepsilon)$:

- The sequence $1, 0.999, 1, 0.999, \cdots$ does not converge to 1 even every term is very close to 1 in the sense that $\forall n, |x_n - 1| < 0.01$.
- The sequence $1, 0, 1, 0, 0, 0, 0, 1, 0, \cdots, 0, 1, 0, \cdots$ does not converge to 0, where the n-th 1 is followed by n^2 zeros. You can see that almost all the terms are 0, but there always exists some 1's beyond any position of the sequence.

How to understand this definition: however small ε is, there is a point in the sequence such that beyond that point, all the terms are within ε of x.

This is a limiting behavior of a sequence and the first few terms do NOT affect the limit, even if they are quite far from the limit. This is the meaning of the **tail sequence** introduced in the text book. You can also refer to the Remark on page 66 of the textbook.

As an exercise, you can try to show that $|x_n - x| < \varepsilon$ can be replaced by $|x_n - x| \le \varepsilon$. It's a convention in our course and textbook that you should also use $<$ in all your assignments and exams.

2. When asked to prove that a given sequence is convergent or to find its limit, there are mainly two cases.

1°. If you are asked to prove **by definition**, then the only tools allowed are elementary algebraic identities, inequalities, mathematical induction and the knowledge we learned in Chapter 2, including Archimedean Property, Bernoulli's inequality, AM-GM inequality and so on. You must **start from the original definition** and no other theorems can be used except otherwise stated. The general procedure is

- Let $\varepsilon > 0$ be arbitrary (**once ε is fixed, it cannot be changed**).
- Find some $K(\varepsilon) \in \mathbb{N}$, which usually depends on our choice of ε.
- Show that for any n larger than this $K(\varepsilon)$, we have $|x_n - x| < \varepsilon$.

The most difficult step is usually how to find a suitable $K(\varepsilon)$. Sometimes it is quite tedious and involves complicated calculations. The usual way is to substitute in x_n, x and then solve this inequality. Let's look at an example to illustrate this procedure.

Q: Show by definition that

$$\lim_{n\to\infty} \frac{5n^2+2n+3}{n^2+n+2} = 5.$$

Given any positive real number ε, we desire to show that there exists $K(\varepsilon) \in \mathbb{N}$ such that $\forall n \geq K(\varepsilon)$ we have

$$\left| \frac{5n^2+2n+3}{n^2+n+2} - 5 \right| < \varepsilon.$$

Then we can solve above inequality to obtain a satisfactory $K(\varepsilon)$:

$$\left| \frac{5n^2+2n+3}{n^2+n+2} - 5 \right| < \varepsilon \Longleftrightarrow \frac{3n+7}{n^2+n+2} < \varepsilon \Longleftrightarrow \varepsilon n^2 + (\varepsilon - 3)n + 2\varepsilon - 7 > 0.$$

However, solutions to this inequality have different formulas depending on various values of ε and can be complicated.

Notice that it suffices to find one $K(\varepsilon)$, we do not need to find out all legal $K(\varepsilon)$. So we can use some basic inequalities and known results to simplify our computations:

$$\begin{aligned}
\left| \frac{5n^2+2n+3}{n^2+n+2} - 5 \right| < \varepsilon \Longleftrightarrow & \frac{3n+7}{n^2+n+2} < \varepsilon \\
\Longleftarrow & \frac{3n+7}{n^2} < \varepsilon \\
\Longleftarrow & \frac{10n}{n^2} = \frac{10}{n} < \varepsilon \\
\Longleftarrow & n \geq \left[\frac{10}{n} \right] + 1 := K(\varepsilon).
\end{aligned}$$

Roughly speaking, we are seeking for a sufficient condition instead of an equivalent condition. Please notice the different use of $\Longleftarrow$ and $\Longleftrightarrow$.

2°. On the other hand, if you are **not required to show by definition**, then any theorems, properties and known limits we have learned in the lectures can be applied and our arguments can be simplified a lot. And you should do enough exercises to familiarize yourself with these theorems.

At this stage of study, **theorem 3.1.10** which can be regarded as an application of the Squeeze Theorem, is of special use:

Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have

$$|x_n - x| \leq Ca_n \quad \text{for all} \quad n \geq m,$$

then it follows that $\lim_{n\to\infty} x_n = x$.